COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **24** (3) (2004) 487–501

TRANSITIVITY AND CONNECTIVITY OF PERMUTATIONS

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To Chantal, that we may stay connected beyond the simple line of time.

Received November 3, 1999
Revised November 5, 2002

It was observed for years, in particular in quantum physics, that the number of connected permutations of $[0;n]$ (also called indecomposable permutations), i.e. those ϕ such that for any $i<n$ there exists $j>i$ with $\phi(j)<i$, equals the number of pointed hypermaps of size n, i.e. the number of transitive pairs (σ,θ) of permutations of a set of cardinality n with a distinguished element.

The paper establishes a natural bijection between the two families. An encoding of maps follows.

1. Introduction

1.1. Preliminary definitions

In the following B will denote a finite set. The group of all the permutations on B (resp. $[1;n]$) will be denoted by $\mathfrak{S}(B)$ (resp. $\mathfrak{S}_n$). Products of permutations are read from right to left, as for usual functions. A *permutation group* on B is a subgroup of the group $\mathfrak{S}(B)$. The permutation group generated by $\pi_1,\dots,\pi_k$ is denoted $\langle\pi_1,\dots,\pi_k\rangle$. Given a permutation group G on B and an element $x\in B$, the *orbit* of x is defined by $G\cdot x=\{y\in B, \exists\pi\in G, \quad \pi(x)=y\}$. Notice that the orbits of G define a partition of B. A permutation group on B with one orbit only *acts transitively* on B. A subset X of B is *stable under the action of* G if, for any $x\in X$ and any $\pi\in G$, we have $\pi(x)\in X$.

A *numbering* of a finite set B is a bijection from B to $[1;|B|]$. We shall denote $\mathrm{Numb}(B)$ the set of all the numberings of B and, for $L\in\mathrm{Numb}(B)$,

Mathematics Subject Classification (2000): 05A19; 05C30

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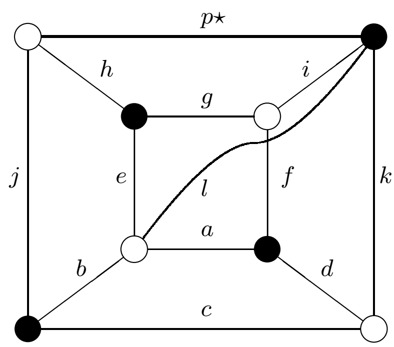


$$\sigma = (a\,b\,e\,l)(c\,d\,k)(f\,g\,i)(h\,j\,p) = \begin{pmatrix} a & b & c & d & e & f & g & h & i & j & k & l & p \\ b & e & d & k & l & g & i & j & f & p & c & a & h \end{pmatrix}$$

$$\theta = (a\,d\,f)(b\,j\,c)(e\,g\,h)(i\,l\,k\,p) = \begin{pmatrix} a & b & c & d & e & f & g & h & i & j & k & l & p \\ d & j & b & f & g & a & h & e & l & c & p & k & i \end{pmatrix}$$

$$r = p$$

Fig. 1. A labeled pointed hypermap.

$x \overset{L}{<} y$ the linear order corresponding to L: $x \overset{L}{<} y$ if $L(x) < L(y)$. We shall use terms as L-greater or L-minimum to refer to the $\overset{L}{<}$ order.

A permutation $\pi \in \mathfrak{S}(B)$ is a *conjugate* of a permutation $\rho \in \mathfrak{S}(B)$ if there exists a permutation $\mu \in \mathfrak{S}(B)$, such that $\pi = \mu\rho\mu^{-1}$. By extension, if $\pi \in \mathfrak{S}(B)$ and $\sigma \in \mathfrak{S}(X)$, where $|X| = |B|$, we say that π is a *conjugate of* σ *in* $\mathfrak{S}(B)$ if there exists a bijection $\mu : B \to X$, such that $\pi = \mu\sigma\mu^{-1}$. For instance, a permutation $\tilde{\pi}$ is a conjugate of π in $\mathfrak{S}_n$ if there exists a numbering L of B, such that $\tilde{\pi} = L\pi L^{-1}$.

Hypermaps generalize the rotation scheme introduced by Heffter [7] and then by Edmonds [6] for encoding a map on an arbitrary orientable surface. A *labeled hypermap* on B is a couple (σ,θ) of permutations on B, such that $\langle\sigma,\theta\rangle$ acts transitively on B. The set B is the *ground set* of the labeled hypermap, elements of B are its *darts*, while its *vertices* and *edges* are the orbits of σ and θ, respectively. The *degree* of a vertex (resp. of an edge) is the size of the corresponding orbit of σ (resp. θ). A dart $b \in B$ is *incident* to a vertex (resp. an edge) if it belongs to the corresponding orbit of σ (resp. θ).

In figures (like Fig. 1), hypermaps are represented by means of their *incidence map*: a bipartite map, whose white nodes (resp. black nodes, resp. arcs) correspond to the vertices (resp. the edges, resp. the darts) of the hypermap. For a discussion about the equivalence of hypermaps and bipartite maps, see [12], [9] and [13]. For hypermap (σ,θ), the clockwise circular order of the arcs around the white nodes (resp. the black nodes) correspond to σ (resp. θ^{-1}).

Two labeled hypermaps (σ_1,θ_1) and (σ_2,θ_2) on B_1 and B_2, respectively are *isomorphic* if there exists an *isomorphism* μ from the labeled hypermap (σ_1,θ_1) to the labeled hypermap (σ_2,θ_2), that is: a bijection $\mu: B_1 \to B_2$, such that $\sigma_2=\mu\sigma_1\mu^{-1}$ and $\theta_2=\mu\theta_1\mu^{-1}$.

A *labeled pointed hypermap* on B is a triple (σ,θ,r), where (σ,θ) is a labeled hypermap on B and $r\in B$ is the *pointed dart*. It defines a *pointed vertex* $\langle\sigma\rangle\cdot r$ and the *pointed vertex degree* $|\langle\sigma\rangle\cdot r|$. A *pointed hypermap* is an equivalence class of the labeled pointed hypermaps for the equivalence relation $\sim$, where $(\sigma_1,\theta_1,r_1)\sim(\sigma_2,\theta_2,r_2)$ if there is an isomorphism μ from the hypermap (σ_1,θ_1) to the hypermap (σ_2,θ_2) which maps r_1 to r_2 (see Fig. 2). A *map* is a hypermap whose edges have all degree two.

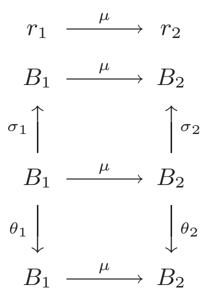


Fig. 2. Equivalence of (σ_1,θ_1,r_1) and (σ_2,θ_2,r_2) by the isomorphism μ.

For a more general discussion about hypermaps, see [3], [8], [11].

Given a numbering L of B, a subset $X\subseteq B$ is an *up-set* of L if, for every $x\in X$ and $y\in B$, $L(y)>L(x)$ implies $y\in X$. Notice that the up-sets of L are totally ordered by inclusion.

Definition 1.1. Let G be a permutation group on B and let L be a numbering of B. The *last connected component* $\mathrm{LCC}(G,L)$ *of the group* G *with respect to* L is the smallest non empty up-set of L stable under the action of G.

When $B=[1;n]$ and L is the identity numbering of $[1;n]$, we shall write $\mathrm{LCC}(G)$ instead of $\mathrm{LCC}(G,L)$. If θ is a permutation on B, we shall write $\mathrm{LCC}(\theta,L)$ and $\mathrm{LCC}(\theta)$ in place of $\mathrm{LCC}(\langle\theta\rangle,L)$ and $\mathrm{LCC}(\langle\theta\rangle)$.

Definition 1.2. Let L be a numbering of a finite set B. A permutation $\theta\in\mathfrak{S}(B)$ is *L-connected* if $\mathrm{LCC}(\theta,L)=B$.

When $B=[1;n]$ and L is the identity, we shall use the term of *connected* instead of L-connected.

Example 1.3.

$$\theta = \begin{pmatrix} 1&2&3&4&5&6 \\ 3&4&1&6&5&2 \end{pmatrix} \text{ is connected;}$$
$$\theta = \begin{pmatrix} 1&2&3&4&5&6 \\ 3&2&1&6&5&4 \end{pmatrix} \text{ is not connected } (\mathrm{LCC}(\theta)=\{4,5,6\}).$$

1.2. Results

We may now express the main theorem of the paper.

In Section 2 (Definition 2.4), we introduce a function ψ which associates a numbering of B with every triplet $(\sigma,\theta,r)\in\mathfrak{S}(B)\times\mathfrak{S}(B)\times B$ such that $\langle\sigma,\theta\rangle$ acts transitively on B. We then prove:

Theorem. *Let θ be a permutation on a finite set B, let $r\in B$ be a distinguished element of B and let $1\le d\le|B|$ be an integer.*

Then, the function $\psi(\cdot,\theta,r)$ is a bijection

- *from the set of the permutations σ on B such that $\langle\sigma,\theta\rangle$ acts transitively on B and $|\langle\sigma\rangle\cdot r|=d$,*
- *to the set of the numberings L of B, such that $L(r)=d$ and $r\in\mathrm{LCC}(\theta,L)$.*

From this theorem, we then deduce the following bijection on pointed hypermaps:

Theorem. *Let $1\le d\le n$ be integers.*

The mapping from the set of the labeled pointed hypermaps with n darts to $\mathfrak{S}_n$ defined by

$$(\sigma,\theta,r)\mapsto\psi(\sigma,\theta,r)\,\theta\,\psi(\sigma,\theta,r)^{-1}$$

induces a bijection

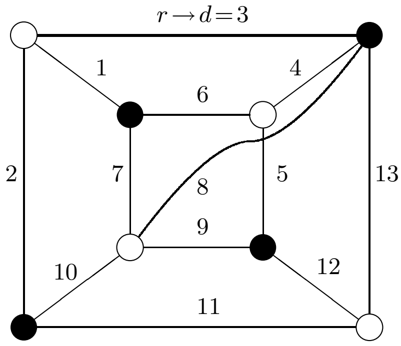
- *from the set of the pointed hypermaps with n darts, with pointed vertex degree d and with a representative of the form (σ,θ,r),*
- *to the set of the conjugates $\tilde\theta$ of θ in $\mathfrak{S}_n$, such that $|\mathrm{LCC}(\tilde\theta)|>n-d$ and $|\langle\tilde\theta\rangle\cdot d|=|\langle\theta\rangle\cdot r|$.*

By a slight transformation, we will deduce:

Theorem. *Let $1\le d\le n$ be positive integers.*

There is a bijection from the set of the pointed hypermaps with n darts and with pointed vertex degree d to the set of the connected permutations $\alpha\in\mathfrak{S}([0;n])$ such that $\alpha^{-1}(0)=d$. (see Fig. 3).

Also:



$$(\sigma, \theta, r) \mapsto (\tilde{\sigma}, \tilde{\theta}, d) \mapsto \alpha$$

$$\tilde{\sigma} = (1\ 2\ 3)\,(4\ 5\ 6)\,(7\ 8\ 9\ 10)\,(11\ 12\ 13)$$
$$(L(r) = d = 3, L(m_1) = 6, L(m_2) = 10, L(m_3) = 13)$$
$$\tilde{\theta} = (1\ 7\ 6)\,(2\ 11\ 10)\,(3\ 4\ 8\ 13)\,(5\ 9\ 12)$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 7 & 11 & 4 & 8 & 9 & 1 & 6 & 13 & 12 & 2 & 10 & 5 & 3 \end{pmatrix}$$
$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 4 & 7 & 11 & 0 & 8 & 9 & 1 & 6 & 13 & 12 & 2 & 10 & 5 & 3 \end{pmatrix}$$

Fig. 3. Bijection between pointed hypermaps and connected permutations.

Theorem. *Let $1 \leq d \leq n$ be positive integers.*

There is a bijection from the pointed maps with m edges (that is: with $2m$ darts) which have pointed degree d to the connected fixed-point free involutions α on $[0; 2m+1]$, such that $\alpha^{-1}(0) = d+1$.

Last, we give algorithms to compute the bijections used in the theorems, for the sake of encoding pointed hypermaps and pointed maps as connected permutations and connected fixed-point free involutions. We also give algorithms for the converse bijections, reconstructing pointed hypermaps and pointed maps from their code.

2. More definitions and basic properties

In the following, B denotes a finite set.

Definition 2.1. Let θ be a permutation on B and let L be a numbering of B. An element $b \in B$ is (θ, L)-*minimal* if

$$\forall b' \overset{L}{\geq} b, \quad \theta(b') \overset{L}{\geq} \theta(b). \tag{1}$$

Definition 2.2. Let σ,θ be permutations on B and let L be a numbering of B. An element $b\in B$ is (σ,θ,L)*-minimal* if

$$\forall b' \in \langle\sigma\rangle \cdot b, \quad \theta(b') \overset{L}{\geq} \theta(b). \tag{2}$$

We shall now introduce two mappings. The first, explicitely and the second, algorithmically.

Definition 2.3 (mapping $\psi^\star$). With a triplet $(L,\theta,r)\in \mathrm{Numb}(B)\times\mathfrak{S}(B)\times B$, we associate the permutation $\sigma=\psi^\star(L,\theta,r)$ on B, defined by:

$$\sigma(b) = \begin{cases} L^{-1}(1) & \text{if } b = r,\\ \text{successor}(\overset{L}{<}, r) & \text{if } b = m_1,\\ \text{successor}(\overset{L}{<}, m_i) & \text{if } b = m_{i+1} (1 \leq i < s),\\ \text{successor}(\overset{L}{<}, b) & \text{otherwise.}\end{cases} \tag{3}$$

where $\text{successor}(\overset{L}{<},x)$ is the successor of x in the linear order $\overset{L}{<}$, and where the elements $m_1 \overset{L}{<} m_2 \overset{L}{<} \dots \overset{L}{<} m_s$ are the (θ,L)-minimal elements of B which are strictly greater than r.

Notice that $\sigma=\psi^\star(L,\theta,r)$ is defined in such a way that the (θ,L)-minimal elements $m_1,\dots,m_s$ are also (σ,θ,L)-minimal.

The search for an inverse mapping leads to the following algorithmic definition:

Definition 2.4 (mapping ψ). With a labeled pointed hypermap (σ,θ,r), we associate the numbering $L = \psi(\sigma,\theta,r)$ of B, defined by the following algorithmic construction:

let $m_0=r$
for $i=1$ to $|\langle\sigma\rangle\cdot m_0|$ **do**
  let $L(\sigma^i(m_0))=i$
end for
$X\leftarrow\langle\sigma\rangle\cdot m_0, \quad s\leftarrow 0$
while $X\neq B$ **do**
  let $m_{s+1}\in B\setminus X$ be such that $\theta(m_{s+1})\in X$ and $L(\theta(m_{s+1}))$ minimum.
  for $i=1$ to $|\langle\sigma\rangle\cdot m_{s+1}|$ **do**
    let $L(\sigma^i(m_{s+1}))=|X|+i$

end for
$X \leftarrow X \cup \langle\sigma\rangle \cdot m_{s+1}, \quad s \leftarrow s+1$
end while

As $\langle\sigma,\theta\rangle$ acts transitively on B, if no element of X has a θ^{-1}-value outside it, then X is an orbit stable under the actions of σ and θ^{-1} and thus includes $\langle\sigma,\theta\rangle \cdot r = B$. Therefore m_{s+1} may be computed as long as X is different from B. Hence the sequence eventually ends with $X = B$.

Lemma 2.1. *Let B be a finite set, let $\theta \in \mathfrak{S}(B)$ and let $r \in B$.*

Then, $\psi(\cdot,\theta,r)$ is a bijection from the set of the pointed labeled hypermaps on B to the subset of Numb(B) *formed by the numberings L such that $\langle\psi^\star(L,\theta,r),\theta\rangle$ acts transitively on B.*

Moreover, the restriction of $\psi^\star(\cdot,\theta,r)$ to the subset of the numberings $L \in$ Numb(B) *such that $\langle\psi^\star(L,\theta,r),\theta\rangle$ acts transitively on B is the mapping inverse to $\psi(\cdot,\theta,r)$.*

Proof. Assume (σ,θ,r) is a labeled pointed hypermap and let $L = \psi(\sigma,\theta,r)$. The darts $m_1,\dots,m_s$ computed by the algorithm of Definition 2.4 are (θ,L)-minimal by construction. Thus, it is easily checked that, according to Definition 2.3, $\psi^\star(L,\theta,r) = \sigma$.

Conversely, assume $L \in \mathrm{Numb}(B)$ is such that $\langle\psi^\star(L,\theta,r),\theta\rangle$ acts transitively on B, and let $\sigma = \psi^\star(L,\theta,r)$. Then, according to Definition 2.3, the elements $m_1,\dots,m_s$ will be (σ,θ,L)-minimal. Thus, they will correspond to the elements (also denoted $m_1,\dots,m_s$) computed by the algorithm of Definition 2.4 and hence $\psi(\sigma,\theta,r) = L$. ∎

Lemma 2.2. *Let G be a permutation group on a finite set B_1, $L \in$* Numb(B_1), *and μ a bijection from B_1 to a set B_2. Then:*

$$\mathrm{LCC}(\mu G \mu^{-1}, L\mu^{-1}) = \mu(\mathrm{LCC}(G,L)),$$
$$\psi(\mu\sigma\mu^{-1}, \mu\theta\mu^{-1}, \mu(r)) = \psi(\sigma,\theta,r)\mu^{-1}.$$

Proof. These equalities express $\mathrm{LCC}(G,L)$ and $\psi(\sigma,\theta,r)$ under a relabelling μ of B_1. ∎

Lemma 2.3. *Let B_1, B_2 be finite sets, let $r_1 \in B_1, r_2 \in B_2, \theta_1 \in \mathfrak{S}(B_1)$ and $\theta_2 \in \mathfrak{S}(B_2)$.*

If θ_2 is a conjugate of θ_1 in $\mathfrak{S}(B_2)$, such that $|\langle\theta_1\rangle \cdot r_1| = |\langle\theta_2\rangle \cdot r_2|$, then there exists a bijection $\mu : B_1 \to B_2$, such that $r_2 = \mu(r_1)$ and $\theta_2 = \mu\theta_1\mu^{-1}$.

Proof. As θ_2 is a conjugate of θ_1 in $\mathfrak{S}(B_2)$, there exits a bijection $\rho: B_1 \to B_2$, such that $\theta_2 = \rho\theta_1\rho^{-1}$. Let $s = \rho^{-1}(r_2)$. As $|\langle\theta_1\rangle \cdot r_1| = |\langle\theta_2\rangle \cdot r_2| = |\langle\theta_1\rangle \cdot s|$, we may define a bijection $\mu: B_1 \to B_2$ by:

$$\begin{aligned} \mu(\theta_1^i(r_1)) &= \rho(\theta_1^i(s)) && (0 \le i < |\langle\theta_1\rangle \cdot r_1|),\\ \mu(\theta_1^i(s)) &= \rho(\theta_1^i(r_1)) && (0 \le i < |\langle\theta_1\rangle \cdot s|),\\ \mu(x) &= \rho(x) && (\text{if } x \notin \langle\theta_1\rangle \cdot \{r_1, s\}). \end{aligned}$$

Then, $\mu(r_1) = \mu(\theta_1^0(r_1)) = \rho(\theta_1^0(s)) = \rho(s) = r_2$, and $\theta_2 = \mu\theta_1\mu^{-1}$, as required. ∎

Lemma 2.4. *If $(\sigma_1, \theta_1, r_1)$ and $(\sigma_2, \theta_2, r_2)$ are representatives of the same pointed hypermap, then*

$$\psi(\sigma_2, \theta_2, r_2)\,\theta_2\,\psi(\sigma_2, \theta_2, r_2)^{-1} = \psi(\sigma_1, \theta_1, r_1)\,\theta_1\,\psi(\sigma_1, \theta_1, r_1)^{-1}. \tag{4}$$

Proof. If $(\sigma_1, \theta_1, r_1)$ and $(\sigma_2, \theta_2, r_2)$ are representatives of the same pointed hypermap, then there exists a bijection μ from the ground set B_1 of $(\sigma_1, \theta_1, r_1)$ to the ground set B_2 of $(\sigma_2, \theta_2, r_2)$, such that:

$$\begin{aligned} r_2 &= \mu(r_1),\\ \sigma_2 &= \mu\sigma_1\mu^{-1},\\ \theta_2 &= \mu\theta_1\mu^{-1}. \end{aligned}$$

Thus:

$$\begin{aligned} \psi(\sigma_2, \theta_2, r_2) &= \psi(\mu\sigma_1\mu^{-1}, \mu\theta_1\mu^{-1}, \mu(r_1))\\ &= \psi(\sigma_1, \theta_1, r_1)\mu^{-1} && \text{(by Lemma 2.2).} \end{aligned}$$

Hence,

$$\begin{aligned} \psi(\sigma_2, \theta_2, r_2)\,\theta_2\,\psi(\sigma_2, \theta_2, r_2)^{-1} &= \psi(\sigma_1, \theta_1, r_1)\,\mu^{-1}\theta_2\mu\,\psi(\sigma_1, \theta_1, r_1)^{-1}\\ &= \psi(\sigma_1, \theta_1, r_1)\,\theta_1\,\psi(\sigma_1, \theta_1, r_1)^{-1}. \end{aligned}$$ ∎

3. Transitivity and connectivity

We shall now prove the main theorem.

Theorem 3.1. *Let B be a finite set, $n = |B|$, $r \in B$, $\theta \in \mathfrak{S}(B)$ and $1 \le d \le n$ an integer.*

Then, $\psi(\cdot, \theta, r)$ is a bijection

- *from the set of the permutations $\sigma \in \mathfrak{S}(B)$, such that $|\langle\sigma\rangle \cdot r| = d$ and such that $\langle\sigma, \theta\rangle$ acts transitively on B,*

– *to the set of the numberings L of B, such that $L(r)=d$ and $r\in \mathrm{LCC}(\theta,L)$.*

Proof. Let $(L,\theta,r)\in \mathrm{Numb}(B)\times\mathfrak{S}(B)\times B$.

Let us prove by contradiction that $\langle\sigma,\theta\rangle$ acts transitively on B if $r\in \mathrm{LCC}(\theta,L)$: Assume $r\in\mathrm{LCC}(\theta,L)$ and $\langle\sigma,\theta\rangle\cdot r\neq B$. Define b and m by

$$b=\min_{\overset{L}{<}}(B\setminus\langle\sigma,\theta\rangle\cdot r),$$
$$m=\theta^{-1}\Big(\min_{\overset{L}{<}}\{\theta(b'),b'\overset{L}{\geq} b\}\Big).$$

Then, m is the L-smallest (θ,L)-minimal element L-greater or equal to b and thus, by Definition 2.3, $m\in\langle\sigma\rangle\cdot b$. If $m\overset{L}{<}b$, then $m\in\langle\sigma,\theta\rangle\cdot r$ and thus $b\in\langle\sigma,\theta\rangle\cdot r$, a contradiction. If $m\overset{L}{\geq}b$, then the set $\{b'\in B,\quad b'\overset{L}{\geq}b\}$ is stable under the action of θ, thus includes $\mathrm{LCC}(\theta,L)$ and hence includes $r\overset{L}{<}b$, a contradiction.

Conversely, we prove by contradiction that $r\in\mathrm{LCC}(\theta,L)$ if $\langle\sigma,\theta\rangle$ acts transitively on B: Assume $\langle\sigma,\theta\rangle$ acts transitively on B and $r\notin\mathrm{LCC}(\theta,L)$. Let $m=\max_{\overset{L}{<}}(B\setminus\mathrm{LCC}(\theta,L))$. As $\mathrm{LCC}(\theta,L)$ is a L-up-set stable under the action of θ, m is a (θ,L)-minimal element, which is L-greater or equal to r. Thus, according to Definition 2.3, $\mathrm{LCC}(\theta,L)=\{b'\in B,b'\overset{L}{>}m\}$ is a union a orbits of σ, and hence stable under the action of $\langle\sigma,\theta\rangle$, a contradiction.

Thus, for any $(L,\theta,r)\in\mathrm{Numb}(B)\times\mathfrak{S}(B)\times B$, $r\in\mathrm{LCC}(\theta,L)$ if and only if $\sigma=\psi^\star(L,\theta,r)$ is such that $\langle\sigma,\theta\rangle$ acts transitively on B. The theorem is hence a consequence of Lemma 2.1. ∎

4. Encoding of pointed maps and hypermaps

We start with a general theorem allowing to encode pointed hypermaps with given edge-permutation signature and pointed vertex degree.

In order to prove this theorem, we first state the following strengthening of Lemma 2.4:

Lemma 4.1. *Let (σ_1,θ_1,r_1) and (σ_2,θ_2,r_2) be labeled pointed hypermaps on B_1 and B_2, respectively.*

If $|\langle\sigma_1\rangle\cdot r_1|=|\langle\sigma_2\rangle\cdot r_2|$, then (σ_1,θ_1,r) and (σ_2,θ_2,r) are representatives of a same pointed hypermap if and only if

$$\psi(\sigma_2,\theta_2,r_2)\,\theta_2\,\psi(\sigma_2,\theta_2,r_2)^{-1}=\psi(\sigma_1,\theta_1,r_1)\,\theta_1\,\psi(\sigma_1,\theta_1,r_1)^{-1}. \tag{5}$$

Proof. The "if" part follows from Lemma 2.4.

Thus, assume Equation (5) holds. Let $\mu = \psi(\sigma_2,\theta_2,r_2)^{-1}\psi(\sigma_1,\theta_1,r_1)$. Then, (5) rewrites as $\theta_2 = \mu\theta_1\mu^{-1}$. Moreover, according to Definition 2.4, $(\psi(\sigma_1,\theta_1,r_1))(r_1) = |\langle\sigma_1\rangle\cdot r_1| = |\langle\sigma_2\rangle\cdot r_2| = (\psi(\sigma_2,\theta_2,r_2))(r_2)$, and thus $\mu(r_1)=r_2$. Hence:

$$\begin{aligned}\psi\left(\mu\sigma_1\mu^{-1},\theta_2,r_2\right) &= \psi\left(\mu\sigma_1\mu^{-1},\mu\theta_1\mu^{-1},\mu(r_1)\right) && (\text{as } \mu\theta_1\mu^{-1}=\theta_2,\ \mu(r_1)=r_2)\\ &= \psi(\sigma_1,\theta_1,r_1)\mu^{-1} && (\text{by Lemma 2.2})\\ &= \psi(\sigma_2,\theta_2,r_2).\end{aligned}$$

According to Lemma 2.1, $\psi(\cdot,\theta_2,r_2)$ is injective, and thus $\mu\sigma_1\mu^{-1}=\sigma_2$ and hence (σ_1,θ_1,r_1) and (σ_2,θ_2,r_2) represents the same pointed hypermap. ∎

Theorem 4.2. *Let $1\le d\le n$ be integers.*

The mapping from the set of the labeled pointed hypermaps with n darts to $\mathfrak{S}_n$ defined by

$$(\sigma,\theta,r) \mapsto \psi(\sigma,\theta,r)\,\theta\,\psi(\sigma,\theta,r)^{-1}$$

induces a bijection

- *from the set of the pointed hypermaps with n darts, with pointed vertex degree d and with a representative of the form (σ,θ,r),*
- *to the set of the conjugates $\tilde\theta$ of θ in $\mathfrak{S}_n$, such that $|\mathrm{LCC}(\tilde\theta)|>n-d$ and $|\langle\tilde\theta\rangle\cdot d|=|\langle\theta\rangle\cdot r|$.*

Proof. First notice that, according to Lemma 4.1, the image of the mapping does not depend on the choice of the representative and that the mapping is injective.

Consider any conjugate $\tilde\theta$ of θ in $\mathfrak{S}_n$, such that $|\mathrm{LCC}(\tilde\theta)| > n-d$ and $|\langle\tilde\theta\rangle\cdot d| = |\langle\theta\rangle\cdot d|$. According to Lemma 2.3, there exists a numbering $L\in \mathrm{Numb}(B)$ (where B is the ground set of the hypermap), such that $L(r)=d$ and $\tilde\theta=L\theta L^{-1}$. Then, as $\mathrm{LCC}(\tilde\theta)$ is an up-set,

$$\begin{aligned}|\mathrm{LCC}(\tilde\theta)| > n-d &\iff d\in\mathrm{LCC}(\tilde\theta)\\ &\iff d\in\mathrm{LCC}(L\theta L^{-1},\mathrm{Id})\\ &\iff L^{-1}(d)\in\mathrm{LCC}(\theta,L) && (\text{according to Lemma 2.2})\\ &\iff r\in\mathrm{LCC}(\theta,L).\end{aligned}$$

Thus, according to Theorem 3.1, there exists $\sigma\in\mathfrak{S}(B)$, such that $|\langle\sigma\rangle\cdot r|=d$, $\langle\sigma,\theta\rangle$ acts transitively on B and $L=\psi(\sigma,\theta,r)$. Hence, there exists a pointed hypermap with representative (σ,θ,r) and with pointed vertex degree d, such that $\tilde\theta=\psi(\sigma,\theta,r)\,\theta\,\psi(\sigma,\theta,r)^{-1}$. This proves the surjectivity of the mapping. ∎

Corollary 4.3. *Let $1 \le d \le n$ be positive integers.*

Then, the mapping $(\sigma,\theta,r) \mapsto \psi(\sigma,\theta,r)\,\theta\,\psi(\sigma,\theta,r)^{-1}$ induces a bijection from the set of the pointed hypermaps with pointed vertex degree d, to the set of the permutations $\tilde{\theta} \in \mathfrak{S}_n$ such that $|\operatorname{LCC}(\tilde{\theta})| > n-d$.

Proof. Consider all the possible conjugation classes of θ. ∎

Theorem 4.4. *Let $1 \le d \le n$ be positive integers and let*

$$F_1 : \quad [\,0\,;\,n\,] \times \mathfrak{S}_n \to \mathfrak{S}([\,0\,;\,n\,])$$

be the mapping defined by

$$F(d,\theta)(x) = \begin{cases} \theta(d) & \text{if } x = 0, \\ 0 & \text{if } x = d, \\ \theta(x) & \text{otherwise.} \end{cases}$$

Then, the mapping

$$(\sigma,\theta,r) \mapsto F_1(|\,\langle\sigma\rangle \cdot r\,|, \psi(\sigma,\theta,r)\,\theta\,\psi(\sigma,\theta,r)^{-1})$$

induces a bijection from the set of the pointed hypermaps with n darts and with pointed vertex degree d, to the set of the connected permutations $\alpha \in \mathfrak{S}([0;n])$ such that $\alpha^{-1}(0) = d$.

Proof. F_1 is a bijection from $[0;n] \times \mathfrak{S}_n$ to $\mathfrak{S}([0;n])$, such that $F_1(d,\theta)$ is connected if and only if $|\operatorname{LCC}(\theta)| > n-d$. Thus, the theorem follows from Corollary 4.3. ∎

Theorem 4.5. *Let $1 \le d \le n$ be positive integers, and*

$$\operatorname{Shift_d} : \quad [\,1\,;\,n\,] \to [\,0\,;\,n+1\,] \setminus \{0, d+1\}$$

be the bijection defined by

$$\operatorname{Shift_d}(x) = \begin{cases} x & \text{if } x \le d, \\ x+1 & \text{otherwise.} \end{cases}$$

Moreover, let

$$F_2 : \quad [\,1\,;\,n\,] \times \mathfrak{S}_n \to \mathfrak{S}([\,0\,;\,n+1\,])$$

be the mapping defined by

$$F_2(d,\theta)(x) = \begin{cases} d+1 & \textit{if } x=0, \\ 0 & \textit{if } x=d, \\ \mathrm{Shift_d}\,\theta\,\mathrm{Shift_d}^{-1}(x) & \textit{otherwise.} \end{cases}$$

Then, the mapping

$$(\sigma,\theta,r) \mapsto F_2(|\,\langle\sigma\rangle\cdot r\,|, \psi(\sigma,\theta,r)\,\theta\,\psi(\sigma,\theta,r)^{-1})$$

induces a bijection from the set of the pointed maps with m edges and with pointed vertex degree d, to the set of the connected fixed point free involutions $\alpha\in\mathfrak{S}([0;2m+1])$, such that $\alpha(0)=d+1$.

Proof. F_2 is a bijection from the set of couples (d,θ), where θ is a fixed point free involution with $|\mathrm{LCC}(\theta))|>n-d$ to the set of the connected fixed point free involutions α with $\alpha(0)=d+1$. The theorem thus follows from Theorem 4.2, by considering any fixed point free involution θ. ∎

5. Counting

Theorem 5.1. *Let B be a finite set, $r\in B$ a distinguished element of B, G a permutation group on B. Let θ_G be a permutation on B having the same orbits as G. Then, the function $\psi(\cdot,\theta_G,r)$ is a bijection*

- *from the set of the permutations $\sigma\in\mathfrak{S}(B)$, such that $\langle\sigma,G\rangle$ acts transitively on B,*
- *to the set of the numberings L of B such that $r\in\mathrm{LCC}(G,L)$.*

Moreover, if $L=\psi(\sigma,\theta_G,r)$, then $L(r)=|\,\langle\sigma\rangle\cdot r\,|$.

Proof. Let θ be any permutation on B having the same orbits as G. Then, for any permutation σ, $\langle\sigma,G\rangle$ acts transitively on B if and only if $\langle\sigma,\theta\rangle$ acts transitively on B. Similarly, for any numbering L on B, we have $\mathrm{LCC}(G,L)=\mathrm{LCC}(\theta,L)$ as $\mathrm{LCC}(G,L)$ is the smallest up-set of L which is an union of orbits of G.

Thus, according to Theorem 3.1, $\psi(\cdot,\theta,r)$ will be a bijection from the set of the permutations $\sigma\in\mathfrak{S}(B)$, such that $|\,\langle\sigma\rangle\cdot r\,|=d$ and such that $\langle\sigma,G\rangle$ acts transitively on B, to the set of the numberings L of B, such that $L(r)=d$ and $r\in\mathrm{LCC}(G,L)$. ∎

Corollary 5.2. *The number of permutations σ such that $\langle\sigma,G\rangle$ acts transitively on B is equal to*

$$\frac{1}{|B|}\sum_{\text{linear order } L}|\operatorname{LCC}(G,L)| = \frac{|N(G)|}{|B|}\sum_{H \text{ conj } G}|\operatorname{LCC}(H,L_0)|,$$

where L_0 is some fixed numbering of B, $N(G)$ denotes the normalizer of G (i.e. the set of the permutations μ, such that $\mu G\mu^{-1}=G$), and where the last summation is done over all the subgroups of $\mathfrak{S}(B)$ which are conjugates of G.

We shall also mention the following corollary of Theorem 4.2:

Corollary 5.3. *The number of pointed hypermaps with m darts, such that the vertex incident to the pointed dart has degree d is equal to*

$$\sum_{i=0}^{d-1} i!\, f(m-i),$$

where $f(i)$ is the number of connected permutations on $[1;i]$.

6. Algorithms

Algorithm 1 computes the numbering $\psi(\sigma,\theta,r)$, or returns an error if $\langle\sigma,\theta\rangle$ does not act transitively on B.

This algorithm looks like a “shortest path” algorithm, what is not so surprising as the numbering $\psi(\sigma,\theta,r)$ may actually be seen as a shortest path order for some valuation of the directed graph with nodes B and with arcs corresponding to the $x\mapsto\sigma(x)$ and $x\mapsto\sigma\theta^{-1}(x)$ transitions.

In case where $B=[1;n]$, Algorithm 1 may be used to actually compute the conjugate $\psi(\sigma,\theta,r)\theta\psi(\sigma,\theta,r)^{-1}$ of θ encoding the pointed hypermap (σ,θ,r).

Algorithm 2 computes a permutation σ on $[1;n]$ associated with a couple (θ,d). Although this algorithm may be applied on any couple (θ,d), the computed permutation σ actually provides a representative of the hypermap (σ,θ,d) having θ as its code.

Algorithm 1 (Encoding): Computes $\psi(\sigma,\theta,r)$.

Require: $\langle\sigma,\theta\rangle$ acts transitively on B and $r\in B$.
Ensure: $\phi=\psi(\sigma,\theta,r)$.

```
for all b∈B do
  φ(b) ← ∞
end for
for i ← 1 to |⟨σ⟩·r| do
  φ(σ^i(r)) ← i
end for
left ← 1,   right ← |⟨σ⟩·r|
while right < |B| do
  while φθ^{-1}φ^{-1}(left) < ∞ do
    if left = right then
      return an error {⟨σ,θ⟩ does not act transitively on B}
    end if
    left ← left + 1
  end while
  b ← θ^{-1}φ^{-1}(left)
  for i ← 1 to |⟨σ⟩·b| do
    φ(σ^i(b)) ← right + i
  end for
  right ← right + |⟨σ⟩·b|
end while
```

Algorithm 2 (Decoding): Computes a permutation $\sigma\in\mathfrak{S}_n$ associated with a couple (θ,d).

Require: θ is a permutation on $[1;n]$ and $1\le d\le n$ is an integer.
Ensure: $\sigma=\mathrm{F}(\theta,d)$.

```
for i ← 1 to n−1 do
  σ(i) ← i+1
end for
right ← n
i ← n−1
while i > d do
  if θ(i) < θ(right) then
    σ(right) ← i+1
    right ← i
  end if
  i ← i−1
end while
σ(right) ← d+1
σ(p) ← 1
```

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